AN ANALYTICAL MODEL FOR PREDICTING EFFECTIVE MAGNETOSTRICTION OF MAGNETOSTRICTIVE COMPOSITES

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An extended double-inclusion model is developed to predict effective magnetostriction of the composites that consist of magnetostrictive particles and a nonmagnetostrictive matrix. The explicit formulas accounting for the inter-particle interaction and matrix-particle interaction under the external magnetic field are obtained for an elastically transverse isotropic magnetostrictive medium. It is shown that the magnetostriction of magnetostrictive composites depends not only on both the volume fraction of the particles and the material properties, but also on shapes and orientations of the ellipsoidal reinforced particles. To demonstrate the applicability of the model, the calculated results are compared both to the available experimental results and to the predictions calculated from other models.

Keywords: Magnetostrictive composites; effective magnetostriction; double-inclusion.

1. Introduction

The magnetostrictive and magnetomechanical properties of rare-earth-iron compounds have been studied. The composites, composed of magnetostrictive particles dispersed in a non-magnetostrictive matrix such as Al, are susceptible to elastic deformation, and constitute a very interesting group of magnetostrictive materials, which can take advantage of each constituent and consequently have superior magnetoelastic effects as compared to conventional magnetostrictive materials.

Recently, a few models have been developed to predict the magnetostriction of the composites. Herbst et al. proposed a simple model for predicting the effective magnetostriction of the SmFe$_2$/Al and SmFe$_2$/Fe composites. This model describes the magnetostriction problem as an elastic problem that depicts an elastically isotropic magnetostrictive sphere embedded in an infinite medium. It is just a proximate model that is insensitive to material constants. Nan developed a more rigorous model based on the Green’s function. However, the predictions of Nan’s model show that the effective magnetostriction ratio $\lambda^*$ for the SmFe$_2$/Fe is smaller than that for the SmFe$_2$/Al, which is not consistent with the experimental data.

In this paper, attentions are given to develop an extended double-inclusion model for the purpose of accurately predicting the magnetostriction of the

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magnetostrictive composites. This model can account for the interaction between particles (inclusion) and the matrix as well as the inter-particle interaction among particles (inclusions). The comparison of the results calculated by using three different models and the experimental data demonstrates that the present double inclusion model is more consistent with the experimental results than other models.

2. Basic Equations

In the cubic magnetostrictive crystals with the giant magnetostrictive effect, the constitutive equations can be described as

\[ \sigma_{ij} = E_{ijkl}(\varepsilon_{kl} - \varepsilon_{kl}^{ms}), \]

where \( \sigma_{ij}, \varepsilon_{ij}, \) and \( \varepsilon_{ij}^{ms} \) are the stress, total strain, and strain, respectively, induced by a magnetostriction that is the function of the magnetic field \( H_i \). \( E_{ijkl} \) are the elastic moduli that depend on \( H_i \) and \( \sigma_{ij} \). The boundary conditions of the composite can be written as

\[ u_i(S) = \varepsilon_{ij}^S x_j, \quad \phi(S) = -H_i^\infty x_i, \]

where \( u_i, \phi \) and \( S \) are the elastic displacement, magnetic potential and external surface of the composite, respectively. The effective magnetostrictive behaviour of an inhomogeneous composite is defined in terms of averaged fields

\[ \langle \sigma_{ij} \rangle = \bar{E}_{ijkl}(\langle \varepsilon_{kl} \rangle - \varepsilon_{kl}^{ms}), \]

where \( \bar{E}_{ijkl} \) and \( \varepsilon_{ij}^{ms} \) are overall elastic moduli and effective magnetostrictive strain, respectively. Since the average field quantities over the volume can be completely determined by the surface data on \( S \),\(^7,8\) which yields \( \langle \varepsilon_{ij} \rangle = \varepsilon_{ij}^\infty \), Eq. (3) can be rewritten as

\[ \langle \sigma_{ij} \rangle = \bar{E}_{ijkl}(\varepsilon_{kl}^\infty - \varepsilon_{kl}^{ms}). \]

To obtain the effective magnetostriction of the magnetostrictive composites, we firstly modified the elastic double-inclusion model to get an extended magnetoelastic double-inclusion model. A double-cell \( \Omega_2 \) is taken for each inclusion as shown in Fig. 1. The double-cell contains the matrix \( \Omega_2 - \Omega_1 \) and the inhomogeneity \( \Omega_1 \).

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**Fig. 1.** Schematic of the double-inclusion model.
Then the double-cell $\Omega_2$ is embedded in an infinite matrix $D$ with elastic moduli $E^0$ while the boundary is subjected to a far field $\varepsilon^\infty$ (see Fig. 1). For convenience, $\Omega_1$ and $\Omega_2 - \Omega_1$ are denoted by $\Gamma_1$ and $\Gamma_2$, respectively, with the volume fractions, $f_1$ and $f_2$, while $f_1 + f_2 = 1$. $E^1$ and $E^2$ are the elastic moduli of the inhomogeneity $\Gamma_1$ and the matrix $\Gamma_2$, respectively. The eigen-strains in $\Gamma_1$ and $\Gamma_2$ are $\varepsilon^*_1$ and $\varepsilon^*_2$, respectively. From the definition of the effective moduli based on the double-inclusion model,\textsuperscript{7-9} we have

$$\langle \sigma_{ij} \rangle = \sigma_{ij}^\infty + \langle \sigma_{ij}^d \rangle = E_{ijkl}(\varepsilon_{kl}^\infty + \langle \varepsilon_{ij}^d \rangle),$$

(5)

where $\langle \sigma_{ij} \rangle$ and $\langle \varepsilon_{ij}^d \rangle$ are the average disturbance stress and strain in the double inhomogeneity, respectively.

From Eq. (4) to Eq. (5), it can be found that $\varepsilon_{ij}^{ms}$ is an average disturbance strain induced by the eigen-strain $\varepsilon_{ij}^{ms}$ in the inhomogeneity.\textsuperscript{7,8} Thus, $\varepsilon_{ij}^{ms}$ can be expressed as

$$\varepsilon_{ij}^{ms} = \langle \varepsilon_{ij}^d \rangle,$$

(6)

where $\langle \varepsilon_{ij}^d \rangle$ is the average disturbance strain with respect to the eigen-strain $\varepsilon_{ij}^{ms}$. The double-cell gives the effective properties of the composite. Based on the concept of equivalent inclusion,\textsuperscript{6} the equivalent eigen-strain $\varepsilon^{**}|_r$ ($r = 1, 2$) due to the presence of inhomogeneity can be defined as\textsuperscript{7-9}

$$E^1_{ijkl}[\varepsilon^\infty_{kl} + S^1_{klmn}(\varepsilon^*_mn|1 + \varepsilon^{**}_mn|1) + (S^2_{klmn} - S^1_{klmn})(\varepsilon^*_mn|2 + \varepsilon^{**}_mn|2) - \varepsilon^*_{kl}|1]$$

$$= E^0_{ijkl}[\varepsilon^\infty_{kl} + (S^1_{klmn} - I_{klmn})(\varepsilon^*_mn|1 + \varepsilon^{**}_mn|1)$$

$$+ (S^2_{klmn} - S^1_{klmn})(\varepsilon^*_mn|2 + \varepsilon^{**}_mn|2)]$$

(7a)

and

$$E^2_{ijkl}\left[\varepsilon^\infty_{kl} + S^2_{klmn}(\varepsilon^*_mn|2 + \varepsilon^{**}_mn|2) + \frac{f_1}{f_2}(S^2_{klmn} - S^1_{klmn})(\varepsilon^*_mn|1 + \varepsilon^{**}_mn|1) - \varepsilon^*_{kl}|2]\right]$$

$$= E^0_{ijkl}\left[\varepsilon^\infty_{kl} + (S^2_{klmn} - I_{klmn})(\varepsilon^*_mn|2 + \varepsilon^{**}_mn|2)$$

$$+ \frac{f_1}{f_2}(S^2_{klmn} - S^1_{klmn})(\varepsilon^*_mn|2 + \varepsilon^{**}_mn|2)\right],$$

(7b)

where $S^1_{ijkl}$ and $S^2_{ijkl}$ are the Eshelby tensors of $\Omega_1$ and $\Omega_2$, respectively. If $\Omega_1$ and $\Omega_2$ are similar and coaxial, there is $S^1_{ijkl} = S^2_{ijkl} = S_{ijkl}$. Therefore, the equivalent eigen-strain $\varepsilon^{**}|_r$ ($r = 1, 2$) are obtained as follows:

$$\varepsilon^{**}|_r = [(E^0_{ijkl} - E^r_{ijkl})^{-1} - S_{ijkl}]^{-1}\varepsilon^\infty_{kl} + [(E^0_{ijkl} - E^r_{ijkl})^{-1} - S_{ijkl}]^{-1}$$

$$\times (S_{klmn} - I_{klmn})\varepsilon^*_mn|r, \quad r = 1, 2,$$

(8)

where $I_{klmn}$ is the fourth-order unit tensor. It is easy to obtain $S_{ijkl}$ when the shape of inhomogeneity is one of some special shapes such as the sphere, the cylinder, and the penny shape crack. Thus, we get the average disturbance field in the double-cell

$$\langle \varepsilon_{ij}^d \rangle = S^2_{ijkl}[f_1(\varepsilon^*_{kl}|1 + \varepsilon^{**}_kl|1) + f_2(\varepsilon^*_kl|2 + \varepsilon^{**}_kl|2)].$$

(9)
Because the pure magnetostriction only results from the external magnetic field, the effective magnetostriction depends on the average disturbance field in the composite. Now it is considered that a volume $V$ of the magnetostrictive composite with a non-magnetostrictive matrix contains $N$ magnetostrictive particles $\Omega_1 (i = 1, \ldots, N)$. The particle reinforcement is made up of micro-crystallites with a cubic symmetry. The local magnetostrictive strains along the crystallographic axes of a microcrystallite can be obtained as

$$
\varepsilon_{ij}^{ms} = \begin{cases} 
\lambda_{100} + \frac{2}{3} \lambda_{111} \left( \alpha_{3i}^2 - \frac{1}{3} \right), & i = j, \\
\frac{3}{2} \alpha_{3i} \alpha_{3j} \lambda_{111}, & i \neq j,
\end{cases}
$$

(10)

where $\lambda_{100}$, $\lambda_{111}$, and $\lambda''$ are the usual magnetostriction constants of the cubic microcrystallite, respectively. $(\alpha_{ij})$ is a transformation tensor that transforms the local crystallographic axes, $X_0^j$, to the material axes, $X_i$.

Since the magnetostrictively induced strain can be considered as an eigen-strain, from Eqs. (7)–(9), we can obtain the average disturbance field of the composite by replacing $\varepsilon_{ij}$ with $\varepsilon_{ij}^{ms}$. In this model, the two-phase composite is considered such that the geometry of the particle reinforcements is spherical with a volume fraction $f$. Only an external magnetic field $H_3$ is applied along the $X_3$ axis of the representative element of the composite material. To simplify the expression, it is considered that $H_3$ is parallel to the axis $<111>$. This is reasonable because the magnetostriction reaches saturation that results from the magnetic domain switching under a larger external field $H_3$.

As shown in Fig. 2, since the particles are randomly distributed in the matrix, the general expression can be obtained for the representative double-cell. Therefore, according to the interaction between particles and the matrix as well as the

![Fig. 2. The Double-cell microstructure.](image-url)
inter-particle interaction among the particles under the external magnetic field, it is reasonable to assume that the far field strain $\varepsilon_{ij}^\infty|_{ms}$ induced by the magnetic force is equal to the average disturbance strain in the double-cell when the external applied stress $\langle \sigma_{ij} \rangle$ is zero and the magnetization is saturated, i.e.

$$\varepsilon_{ij}^\infty|_{ms} = \langle \varepsilon_{ij}^d \rangle.$$  \hfill (11)

Because the matrix is nonmagnetostrictive, $\varepsilon_{ij}^*|_2$ is zero while $\varepsilon_{ij}^*|_1 = \varepsilon_{ij}^{ms}$, Eq. (11) can be rewritten as

$$\langle \varepsilon_{ij}^d \rangle = f S_{ijmn} \{ Q_{mnrs}(S_{rscpq} - I_{rscpq}) + I_{mnpq} \} \varepsilon_{rr}^{ms} + f S_{ijmn} Q_{mnrs} \langle \varepsilon_{rs}^d \rangle$$  \hfill (12)

with the $Q_{ijkl}$ defined by

$$Q_{ijkl} = [(E_{ijmn} - E_{ijmn}^1) E_{mnkl}^0 - S_{ijkl}]^{-1},$$  \hfill (13)

where $\varepsilon_{ij}^{ms}$ is the magnetostrictively induced strain of the particle reinforcement, which is defined in Eq. (10). From Eq. (12), the effective magnetostrictive strain field can be obtained

$$\varepsilon_{ij}^{ms} = \langle \varepsilon_{ij}^d \rangle = f(I_{ijkl} - f S_{ijmn} Q_{mnkl})^{-1}[S_{klrs} + S_{klmn} Q_{mpnq}(S_{pars} - I_{pqrs})] \varepsilon_{rs}^{ms}. \hfill (14)$$

According to the definition of magnetostriction, the effective magnetostriction can be written as

$$\lambda_s = \frac{2}{3}(\langle \varepsilon_\parallel \rangle - \langle \varepsilon_\perp \rangle), \hfill (15)$$

where $\langle \varepsilon_\parallel \rangle$ and $\langle \varepsilon_\perp \rangle$ are macroscopically magnetostrictive strains parallel and perpendicular to the applied magnetic field $H_3$, respectively. When the matrix is isotropic or transversely isotropic, the Eshelby tensors can be obtained analytically for the particles with an elliptical shape and numerically for those with other shapes. According to Eq. (12), the effective magnetostriction depends on the properties of the matrix and the shape of reinforced particles. In this case, $\langle \varepsilon_\parallel \rangle$ and $\langle \varepsilon_\perp \rangle$ can be expressed as

$$\langle \varepsilon_\parallel \rangle = \langle \varepsilon_{33} \rangle, \quad \langle \varepsilon_\perp \rangle = \langle \varepsilon_{11} \rangle = \langle \varepsilon_{22} \rangle. \hfill (16)$$

The effective normalized magnetostriction $\lambda^*$ can be obtained easily

$$\lambda^* = \frac{\lambda_s(f)}{\lambda_s(f = 1)}. \hfill (17)$$

3. Results and Discussion

The properties of the metal matrix and the SmFe$_2$ reinforced particles are presented in Table 1. Note that the metal matrix is elastically isotropic while the reinforced SmFe$_2$ particles are not elastically isotropic. Furthermore, it is assumed
Table 1. Properties of SmFe$_2$ particles and the metal matrix Al, Fe$^{5,7}$.

<table>
<thead>
<tr>
<th></th>
<th>Young’s modulus $E_1$ (GPa)</th>
<th>Poisson’s ratio $\nu_1$</th>
<th>$E_{12}$ (GPa)</th>
<th>$E_{44}$ (GPa)</th>
<th>$\lambda_{11}$ (ppm)</th>
<th>$\lambda_{100}$ (ppm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>SmFe$_2$</td>
<td>82</td>
<td>66</td>
<td>22</td>
<td>$-2100$</td>
<td>$-700$</td>
<td></td>
</tr>
<tr>
<td>Al</td>
<td>62</td>
<td>0.33</td>
<td>92</td>
<td>45.2</td>
<td>23.3</td>
<td></td>
</tr>
<tr>
<td>Fe</td>
<td>208</td>
<td>0.3</td>
<td>280</td>
<td>120</td>
<td>80</td>
<td></td>
</tr>
</tbody>
</table>

Fig. 3. Comparison of the effective saturation magnetostriction ratio $\lambda^*$ of SmFe$_2$/Fe magnetostrictive composites calculated by the extended double-inclusion model (denoted by DI model) and experimental results. The circle dots are experimental data of SmFe$_2$/Fe magnetostrictive composites (Herbst, Capehart and Pinkerton, 1997). The predictions from the Nan’s model (denoted by Nan’s) and the Herbst’s model (denoted by HCP) are presented too.

Fig. 4. Comparison of the effective saturation magnetostriction ratio $\lambda^*$ calculated by using three models and the experimental results for the SmFe$_2$/Al magnetostrictive composites. The triangle dots are experimental data of the SmFe$_2$/Al magnetostrictive composites (Herbst, Capehart and Pinkerton, 1997).
that the reinforced particles are elliptical. Figures 3 and 4 show the variation of effective saturation magnetostriction ratio $\lambda^*$ of the SmFe$_2$/Al and SmFe$_2$/Fe magnetostriction composites with the change of the filled volume fraction $f$ of the magnetostrictive particles, respectively. The $\lambda^*$ values predicted for the SmFe$_2$/Al and SmFe$_2$/Fe composites by use of the proposed model are very consistent with the experimental data. The HCP model predicts the same results for these two composites. The predictions of the $\lambda^*$ values for the SmFe$_2$/Fe composite by using Nan’s model are smaller than the experimental results (shown in Fig. 3), but those for the SmFe$_2$/Al composite are larger than the experimental results (shown in Fig. 4). Furthermore, the Nan’s model shows that the effective saturation magnetostriction ratio $\lambda^*$ predicted for the SmFe$_2$/Fe is smaller than the $\lambda^*$ values predicted for the SmFe$_2$/Al. This is because the Nan’s model only takes account of the elastic interaction between the particles and the matrix. However, the effective saturation magnetostriction ratio $\lambda^*$ predicted for the SmFe$_2$/Fe by using the proposed model is larger than that predicted for the SmFe$_2$/Al because the inter-particle interaction induced by the external magnetic field is significant, which is consistent with the experimental results.

4. Concluding Remarks

An extended double-inclusion model is developed to predict effective magnetostriction of the composites that consist of magnetostrictive particles and a nonmagnetostrictive matrix. The explicit formulas accounting for the inter-particle interaction and matrix-particle interaction under the external magnetic field are obtained for an elastically transverse isotropic magnetostrictive medium. It is shown that the magnetostriction of magnetostrictive composites depends not only on the volume fraction of the particles, the material properties, but also on the shapes and orientations of the ellipsoidal inhomogeneities. The comparison of the results calculated by using three different models and the experimental data demonstrates that the proposed double inclusion model is more consistent with the experimental results than other models.

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