The effective Young’s modulus of composites beyond the Voigt estimation due to the Poisson effect

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The Voigt estimation or the rule of mixture has been believed to be the upper bound of the effective Young’s modulus of composites. However, this is only true in the situations where the Poisson effect is not significant. In this paper, we accurately derived the effective compliance matrix for two-phase layered composites by accounting for the Poisson effect. It is interesting to find that the effective Young’s modulus in both transverse and longitudinal direction can exceed not only the Voigt estimation, but also the Young’s modulus of the stiffest constituent phase. Moreover, the longitudinal (or parallel connection) Young’s modulus is not always larger than the transverse (or serial connection) one. For isotropic composites, it has also been demonstrated that the Voigt estimation is not the upper bound for the effective Young’s modulus. Therefore, one should be careful in applying the well known bound estimations on the effective Young’s modulus of composites if one of the phases is near its incompressibility limit.

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1. Introduction

The upper and lower bounds for the effective stiffness of two-phase composites have been studied for a long time. Among these works, Voigt [1] adopted isostrain assumption to obtain the estimation of the effective composite stiffness matrix as the weighted volume average of the stiffness matrices of constituent phases. Based on the isostrain assumption, Reuss [2] estimated the effective composite compliance matrix as the weighted volume average of the compliance matrices of constituent phases. Hill [3] has proven that for isotropic constituent phases and composites, Voigt estimation and Reuss estimation, respectively, provide the upper and lower bounds for the effective bulk and shear moduli of composites.

Although the upper and lower bounds of the effective Young’s modulus of composites are not given explicitly in previous works, two special composite layouts (serial connection as shown in Fig. 1 and parallel connection as shown in Fig. 2) are always used in many textbooks and literatures (e.g., Ref. [4]) to investigate these bounds. The constituent phases are isotropic elastic with Young’s modulus $E_A$ and $E_B$, volume fractions $\phi_A$ and $\phi_B$, respectively.

By neglecting the Poisson effects, the serial and parallel layouts are essentially one-dimensional models, and satisfy the isostrain (Voigt) and isostress (Reuss) conditions. Based on Hill’s work [3], the bounds for the Young’s modulus of the composite $E_{\text{composite}}$ therefore can be given as

$$\frac{E_{\text{Reuss}}}{E_A} = \frac{\phi_A}{\phi_B}, \quad \text{or} \quad E_{\text{Reuss}} = \frac{E_A E_B}{\phi_B E_A + \phi_A E_B},$$

and

$$E_{\text{Voigt}} = \phi_A E_A + \phi_B E_B.$$  \hspace{1cm} (3)

Here, the overhead tildes in $E_{\text{Reuss}}$ and $E_{\text{Voigt}}$ mean that these effective moduli only approximately satisfy Reuss (isostress) and Voigt (isostrain) conditions due to the neglecting of the Poisson effect.

Based on Eqs. (1)–(3), the following inequalities can be derived and have been widely considered valid for any situation.

Inequality I

$$E_{\text{composite}} \leq E_{\text{Voigt}} = \phi_A E_A + \phi_B E_B.$$  \hspace{1cm} (4)

implies that the approximate Voigt estimation $E_{\text{Voigt}}$, i.e., the weighted volume average of the Young’s modulus of constituent phases, can be used as the upper bound of the effective Young’s modulus of the composite.

Inequality II

$$E_{\text{eff}} = E_{\text{parallel}} = E_{\text{serial}} \leq T_{\text{parallel}} = T_{\text{serial}} = E_{\text{eff}}^\text{layered}.$$  \hspace{1cm} (5)

implies that the longitudinal (or parallel) stiffness of a layered composite $E_{\text{eff}}^\text{layered}$ is always larger than the transverse (or serial) stiffness $E_{\text{eff}}^\text{layered}$ (see Figs. 1 and 2).

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In this paper, a two-phase layered composite as shown in Fig. 1, the stress and the strain in each phase of RVE are uniform, and they should satisfy the following equations:

**Constitutive equations:**

\[
\begin{align*}
\sigma_x &= \frac{1}{E_A} (\sigma_x - \nu_y \sigma_y - \nu_z \sigma_z), \\
\sigma_y &= \frac{1}{E_A} (\sigma_y - \nu_x \sigma_x - \nu_z \sigma_z), \\
\sigma_z &= \frac{1}{E_A} (\sigma_z - \nu_x \sigma_x - \nu_y \sigma_y), \\
\epsilon_x &= \frac{1}{E_A} (\epsilon_x - \nu_y \epsilon_y - \nu_z \epsilon_z), \\
\epsilon_y &= \frac{1}{E_A} (\epsilon_y - \nu_x \epsilon_x - \nu_z \epsilon_z), \\
\epsilon_z &= \frac{1}{E_A} (\epsilon_z - \nu_x \epsilon_x - \nu_y \epsilon_y).
\end{align*}
\]

(9)–(13)

Here, \(\sigma\) and \(\epsilon\) are stress and strain; \(E\) and \(\nu\) are Young’s modulus and Poisson ratio; the superscripts or subscripts “A” and “B” denote phase A and phase B, respectively. It is noted that there is no shear stress and strain in this situation, so they are not included in the equations above.

**Equilibrium equations:**

\[
\begin{align*}
\bar{\sigma}_x &= \frac{h_A \sigma_x^A + h_B \sigma_x^B}{h_A + h_B} = 0, \\
\bar{\sigma}_y &= \frac{h_A \sigma_y^A + h_B \sigma_y^B}{h_A + h_B} = 0, \\
\bar{\sigma}_z &= \frac{h_A \sigma_z^A + h_B \sigma_z^B}{h_A + h_B} = 0.
\end{align*}
\]

(15)–(17)

Here, \(h_A\) and \(h_B\) are the layer thickness of phase A and phase B (see Fig. 1), respectively.

**Kinematic equations:**

\[
\begin{align*}
\bar{\epsilon}_x &= \epsilon_x^A = \epsilon_x^B, \\
\bar{\epsilon}_y &= \epsilon_y^A = \epsilon_y^B, \\
\bar{\epsilon}_z &= \epsilon_z^A h_A + \epsilon_z^B h_B. \\
\end{align*}
\]

(18)–(20)

By solving Eqs. (9)–(20), we can compute the Young’s modulus along z-direction (or transverse direction) as

\[
E_{\text{eff}}^z = \frac{1}{E_{\text{symm}}} = \frac{1}{\epsilon_z} = \frac{E_A E_B}{\Phi_A E_A + \Phi_B E_B - \frac{\Phi_B \nu_B V_B h_A - \Phi_A \nu_A V_A h_B}{h_A + h_B}},
\]

(21)

where \(\Phi_A = h_A/(h_A + h_B)\) and \(\Phi_B = h_B/(h_A + h_B)\) are the volume fractions of phase A and phase B, respectively. It should be pointed out that all derivations in this paper have been checked by the mathematical software Maple. Another compliance component can also be obtained from this loading situation

\[
S_{15}^\text{eff} = \bar{\epsilon}_x = \frac{\Phi_A V_A + \Phi_B V_B - V_A V_B}{\Phi_A V_A E_A + \Phi_B V_B E_B - \Phi_B E_A - \Phi_A E_B}.
\]

(22)

**Remark 1.** The assumptions used in the Reuss (isostress) approximation are essentially \(\sigma_x^A = \sigma_y^A = 0\) and \(\sigma_z^B = \sigma_y^B = 0\), which are too
strong and are therefore discarded in the derivation above. Instead, by considering the mutual confinement between layers, $e^{1}_x = e^{2}_y$ and $e^{2}_y = e^{1}_z$ are used to determine the effective moduli.

Although two phases are connected in a serial way along the loading direction (see Fig. 1), it is found that the Reuss estimation of $z$-direction Young’s modulus Eq. (2) is very different from the accurate Eq. (21) not only in expression but also in magnitude sometimes. For example, if $v_A = 0.25$, $v_B = 0.5$, $\phi_B = 0.5$ and $E_B/E_A = 0.001$, which are typical material constants for protein and mineral phases in biocomposites, $E_{eff}^z$ from Eq. (21) will be about 286 times of $E_{Reuss}$. This dramatic stiffness increase can be understood by a simple and extreme example. The effective stiffness of an elastic layer (phase $B$) confined between two rigid blocks has the effective stiffness $E_{eff}^{yz} = \frac{E_A E_B}{E_A + \phi_B E_B}$, irrespective of the Young’s modulus of the confined material, the effective stiffness of the layer could be infinity if the material is incompressible $v_B \rightarrow 1/2$. The lateral confinement essentially converts the uniaxial deformation to triaxial deformation.

To validate the above theoretically prediction on the transverse stiffness of layered composites, a uniaxial compression test along the transverse direction was carried out. Fig. 3a shows the experimental setup. The layered composite specimen is made of 0.48 mm thick copper layers and 1.10 mm thick PDMS layers. The experimental setup. The layered composite under longitudinal tension (only $\sigma_x \neq 0$)

Similarly, when the layered composite is subject to longitudinal tension as shown in Fig. 2, the stress and the strain in each phase of RVE are also uniform, but the equilibrium equations and kinematic equations are different from the previous transverse compression case.

\begin{align}
\text{Equilibrium equations:} \\
\sigma_x &= \frac{\phi_A E_A \varepsilon_A + \phi_B E_B \varepsilon_B}{\phi_A + \phi_B}, \\
\sigma_y &= \frac{\phi_A E_A \varepsilon_A + \phi_B E_B \varepsilon_B}{\phi_A + \phi_B} = 0, \\
\sigma_z &= \sigma_x = \sigma_y = 0.
\end{align}

\begin{align}
\text{Kinematic equations:} \\
\varepsilon_x &= e^{1}_x = e^{2}_y, \\
\varepsilon_y &= e^{2}_y = e^{1}_z, \\
\varepsilon_z &= \frac{e^{1}_x E_A + e^{2}_y h_B}{h_A + h_B}.
\end{align}

By solving Eqs. (9)–(14), (23)–(28), the Young’s modulus along $x$-direction (or longitudinal direction) can be determined as

\begin{align}
E_{eff}^{xx} = \frac{1}{S_{11}^{xx}} = \frac{\sigma_x}{\varepsilon_x} = \frac{\phi_A E_A \varepsilon_A + \phi_B E_B \varepsilon_B}{\phi_A E_A (1 - v_B^2) + \phi_B E_B (1 - v_A^2)}.
\end{align}

Moreover, the compliance component $S_{12}$ can also be obtained from this loading condition.

\begin{align}
S_{12}^{yy} = S_{21}^{yy} = \frac{\varepsilon_y}{\sigma_x} = \frac{\phi_A E_A \varepsilon_A + \phi_B E_B \varepsilon_B}{(\phi_A E_A + \phi_B E_B)^2 - (\phi_A E_A + \phi_B E_B) v_A v_B E_B}.
\end{align}

Remark 2. The assumptions used in the Voigt approximation is essentially $\sigma_x = \sigma_y = 0$, which is not realistic. Instead, by considering the mutual confinement between layers, $e^{1}_x = e^{2}_y$ is used to determine the effective moduli.

2.3. The layered composite under shear loading (only $\tau_{xz} \neq 0$)

For the RVE subject shear loading $\tau_{xz}$ as shown in Fig. 4, it is easy to find that $\tau_{xz}$ is the only non-zero stress component in the constituent phases and the composite. The corresponding governing equations are:

\begin{align}
\text{Constitutive equations:} \\
\tau_{xz}^{A} &= \frac{1}{G_{A}} \varepsilon_{xz}^{A}, \\
\tau_{xz}^{B} &= \frac{1}{G_{B}} \varepsilon_{xz}^{B},
\end{align}

where $G_A = \frac{E_A}{2(1 + v_A)}$ and $G_B = \frac{E_B}{2(1 + v_B)}$ are the shear modulus of phase $A$ and phase $B$, respectively.
Equilibrium equations:

\[ \tau_{xz} = \tau_{xz}^A = \tau_{xz}^B. \]  
(33)

Kinematic equations:

\[ \gamma_{xz} = \gamma_{xz}^{A} + \gamma_{xz}^{B} \frac{h_{A} + h_{B}}{h_{A} + h_{B}}. \]  
(34)

By solving Eqs. (31)-(34), the x-z plane effective shear modulus can be determined as

\[ G_{xz}^{eff} = \frac{1}{s_{xz}^{eff}} = \frac{1}{2(s_{11}^{eff} - s_{12}^{eff})} = \Phi_{A}G_{A} + \Phi_{B}G_{B}, \]  
(35)

which is just the Reuss estimation of the effective shear modulus for the composite since the isostress condition is exactly satisfied.

Up to now, all five independent elastic constants \( (S_{11}^{eff}, S_{12}^{eff}, S_{33}^{eff}, S_{13}^{eff}, S_{22}^{eff}) \) have been determined from Eqs. (21), (22), (29), (30), and (35), respectively. Moreover, according to Eq. (8a-d), the x-y plane effective shear modulus can be computed as

\[ C_{xy}^{eff} = \frac{1}{s_{xy}^{eff}} = \frac{1}{2(s_{11}^{eff} - s_{12}^{eff})} = \Phi_{A}C_{A} + \Phi_{B}C_{B}, \]  
(36)

which is just the Voigt estimation of the effective shear modulus for the composite since the isostress condition is exactly satisfied.

It should be emphasized that the effective moduli from above derivation are accurate as long as the following two conditions are satisfied:

(i) the layers are perfectly bonded to each other;
(ii) the thickness of each layer is much smaller than the other two dimensions, say by two orders of magnitude, such that the shear lag edge effect at edges can be ignored.

3. The bounds for the effective Young’s modulus of two-phase anisotropic composites

In the following, we use the above formulas for the effective modulus of layered composite to investigate the bounds of Young’s modulus and check the validness of those inequalities (4)-(6).

For a composite with the following set of parameters: \( \nu_{A} = 0.25, \ E_{B}/E_{A} = 0.5, \ \Phi_{A} = 50\% \), Fig. 5 shows the variations of the normalized Young’s moduli, \( E_{B}/E_{A} \) and \( E_{eff}/E_{A} \), as functions of the Poisson ratio of phase \( B, \nu_{B} \). It is found that when \( \nu_{B} \) approaches 0.5, both longitudinal and transverse Young’s moduli, \( E_{eff}^{eff} \) and \( E_{eff}^{eff} \) can be larger than the approximate Voigt bound \( E_{Voigt} = \Phi_{A}E_{A} + \Phi_{B}E_{B} \). Therefore, the following conclusions can be reached.

**Conclusion 1.** The approximate Voigt estimation, i.e., the weighted volume average of the Young’s modulus of constituent phases, is not always the upper bounds of the Young’s modulus if the Poisson ratio effect is taken into account. In other words, \( Inequality \ I \ (E_{composite}^{eff} \leq E_{Voigt} = \Phi_{A}E_{A} + \Phi_{B}E_{B}) \) may be wrong.

Moreover, the underestimation on the Young’s modulus upper bound by the approximate Voigt estimation can be significant. If we assume that both Poisson ratios can range from 0 to 0.5, the transverse Young’s modulus \( E_{B}^{eff} \) can be 50% larger than \( E_{Voigt} \) when \( \nu_{A} = 0, \nu_{B} = 0.5, \Phi_{A} = 33.3\% \) and \( E_{B}/E_{A} \to 0 \), while the longitudinal Young’s modulus \( E_{A}^{eff} \) can be 7.2% larger than \( E_{Voigt} \) when \( \nu_{A} = 0, \nu_{B} = 0.5, \) and \( E_{B}/E_{A} \to \sqrt{3}/\sqrt{2} \).

Still from Fig. 5, it is found that the transverse stiffness of the layered composite can be larger than its longitudinal stiffness when \( \nu_{B} \) approaches 0.5. We therefore draw the following conclusion.

**Conclusion 2.** The transverse stiffness of the layered composite is not always larger than its longitudinal stiffness if the Poisson ratio effect is taken into account. In other words, \( Inequality \ II \ (E_{eff}^{eff} \leq E_{Voigt} = \Phi_{A}E_{A} + \Phi_{B}E_{B}) \) may be wrong.

Moreover, when \( \nu_{A} = 0, \nu_{B} = 0.5, \Phi_{A} = 33.3\% \) and \( E_{B}/E_{A} \to 0 \), the transverse stiffness \( E_{B}^{eff} \) can be 50% larger than its counterpart in longitudinal direction \( E_{eff}^{eff} \). Therefore, the invalidation of the Inequality II is obvious in this case.

We further investigate the validness of Inequality III. For a layered composite with \( \nu_{A} = 0.25, \ E_{B}/E_{A} = 0.9, \ \Phi_{B} = 50\% \), Fig. 6 shows the variation of the normalized Young’s modulus, \( E_{eff}^{eff}/E_{A} \), as a function of the Poisson ratio of phase \( B, \nu_{B} \). It can also be found that the effective Young’s modulus can exceed the maximum modulus of its constituents when \( \nu_{B} \) approaches 0.5. Therefore, we have the following conclusion.

**Conclusion 3.** The effective Young’s modulus of composites is not always lower than the maximum modulus of its constituents if the Poisson ratio effect is taken into account. In other words, \( Inequality \ III \ (E_{composite}^{eff} \leq \max (E_{A}, E_{B}) \) may be wrong.

In addition, when \( \nu_{A} = 0, \nu_{B} = 0.5, \Phi_{B} = 41.4\% \) and \( E_{B}/E_{A} = 1 \), the transverse stiffness \( E_{B}^{eff} \) can be 20.7% larger than \( \max (E_{A}, E_{B}) \), while the longitudinal stiffness \( E_{A}^{eff} \) can be 7.1% larger than \( \max (E_{A}, E_{B}) \). In this case, the invalidation of Inequality III is obvious.

It is interesting to note from Fig. 5 and Eq. (29) that the approximate Voigt estimation \( E_{Voigt} \) is the lower bound of the longitudinal Young’s modulus \( E_{A}^{eff} \) for layered composites, and this bound can be reached if the Poisson ratios have the following relation:

\[ \nu_{A} = \nu_{B}. \]  
(37)

In this case, the mutual confinement between the two phases actually vanishes. Therefore, there is no other non-zero stress component except \( \sigma_{y} \) and \( \sigma_{z} \), and each phase deforms independently. This is just the assumption in deriving the approximate Voigt estimation \( E_{Voigt} \). However, if \( \nu_{A} \neq \nu_{B} \), the mutual confinement would lead

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**Fig. 4.** A schematic diagram of a layer composite under shear.

**Fig. 5.** The normalized effective Young’s moduli of the layered composite as a function of the Poisson ratio of phase B.
to more non-zero stress components and the higher elastic energy, therefore the longitudinal Young's modulus $E_x^{\text{eff}}$ would become larger.

Similarly, from Fig. 5 and Eq. (21) we find that the approximate Reuss estimation $E_{\text{Reuss}}$ is the lower bound of the transverse Young's modulus $E_x^{\text{eff}}$ for layered composites, and this bound can be reached if the Poisson ratios satisfy the following relation:

$$\frac{v_B}{E_B} = \frac{v_A}{E_A},$$

which also corresponds to no mutual confinement between the two phases.

Moreover, we can prove that $E_{\text{Reuss}} = E_xE_B/(\Phi_A E_A + \Phi_B E_B)$ is really a lower bound for any two-phase composite. Hill's work [3] does not explicitly provide the bounds of the effective Young's modulus along a specific direction in anisotropic composites. Instead, it can be easy to know from his work that the effective compliance matrix of the two-phase composite $S^{\text{eff}}$ must make the matrix $(S) - S^{\text{eff}} = S_1\Phi_A + S_2\Phi_B - S^{\text{eff}}$ positive semi-definite. Therefore, the diagonal components of $S_1\Phi_A + S_2\Phi_B - S^{\text{eff}}$ must be negative. Without losing generality, we only investigate the effective Young's modulus along 1-direction, and

$$S_{11}^{\text{eff}} \Phi_A + S_{12}^{\text{eff}} \Phi_B - S_{11}^{\text{eff}} = \frac{\Phi_A}{E_{11}} + \frac{\Phi_B}{E_{11}} - \frac{1}{E_{11}^{\text{Reuss}}} = \frac{1}{E_{11}^{\text{Reuss}}} - \frac{1}{E_{11}^{\text{Reuss}}} \geq 0,$$

or

$$E_{11}^{\text{Reuss}} \geq \frac{1}{E_{11}^{\text{Reuss}}},$$

which implies that the Young's modulus in any direction cannot be smaller than approximate Reuss estimation. It should be pointed out that the derivation in this paragraph is not limited to layered composites, and the Reuss estimation is therefore a real lower bound. However, from the positive semi-definiteness of $(C) - C^{\text{eff}} = C_1\Phi_A + C_2\Phi_B - C^{\text{eff}}$, i.e., the difference between the stiffness matrices, (see Hill's work), the upper bound of Young's modulus is difficult to obtain, and is absolutely not the approximate Voigt estimation $E_{\text{Voigt}}$ based on the results in this paper.

For comparison, we also investigate the bounds of the shear and bulk moduli. According to Hill's work [3], the Voigt and Reuss estimation are the upper and lower bounds, respectively, for the shear and bulk moduli of isotropic composites, i.e.,

$$\frac{G_A G_B}{\Phi_A G_B + \Phi_B G_A} = G_{\text{Reuss}} \leq G^{\text{iso}} \leq G_{\text{Voigt}} = \Phi_A G_A + \Phi_B G_B,$$

and

$$\frac{K_A K_B}{\Phi_A K_B + \Phi_B K_A} = K_{\text{Reuss}} \leq K^{\text{iso}} \leq K_{\text{Voigt}} = \Phi_A K_A + \Phi_B K_B,$$

where $G$ and $K$ are the shear modulus and bulk modulus, respectively. For the anisotropic composites, such as the layered composites studied in this paper, the effective shear modulus in a specific direction is $G_{ij}^{\text{eff}} = \tau_{ij}/\delta_{ij} = 1/S_{ij}^{\text{eff}} = C_{ij}^{\text{eff}}$, which should satisfy the positive semi-definite requirement for the matrices $S_i\Phi_A + S_j\Phi_B - S^{\text{eff}}$ and $C_i\Phi_A + C_j\Phi_B - C^{\text{eff}}$. Therefore, the Voigt and Reuss estimations are still the bounds for the effective shear modulus, as given in Eqs. (35) and (36). However, due to the hydrostatic stress state, the effective bulk modulus of anisotropic composites depends on multiple components of the compliance matrix, and can be computed from Eq. (7) as

$$K^{\text{eff}} = \frac{1}{\sum_{i,j=1}^{3} \delta_{ij}^{\text{eff}}}.$$

Substituting Eqs. (21), (22), (29), (30) into the above equation, it can be proved that the Voigt and Reuss estimations are the bounds for the effective bulk modulus.

4. The bounds for the effective Young's modulus of two-phase isotropic composites

The above section has shown that the effective Young's modulus of a two-phase layered composite in a specific direction can exceed the approximate Voigt estimation $E_{\text{Voigt}}$. Another question immediately emerges: is there any composite whose effective Young's moduli in all direction is larger than the approximate Voigt estimation $E_{\text{Voigt}}$? To answer this question, we will focus our attention on the composites with isotropic elasticity. Two types of composites are discussed.

4.1. The upper bound of the effective Young's modulus for isotropic layered composites

The two-phase layered composites studied in previous sections are usually transversely isotropic and have five independent elastic constants (e.g., $S_{33}^{\text{eff}}, S_{13}^{\text{eff}}, S_{11}^{\text{eff}}, S_{12}^{\text{eff}}$ and $S_6^{\text{eff}}$), while the isotropic elastic materials have only two independent elastic constants (e.g., $E^{\text{eff}}$ and $\nu^{\text{eff}}$). Therefore, the layered composite may become isotropically elastic only if the following three relations among effective moduli are satisfied

$$G_{ij}^{\text{eff}} = G_{ij}^{\text{iso}},$$

$$E_x^{\text{eff}} = E_x^{\text{iso}},$$

$$S_6^{\text{eff}} = S_6^{\text{iso}}.$$

Substituting Eqs. (35) and (36) into Eq. (44), we obtain one necessary condition for the elastic isotropy of two-phase layered composites

$$G_A = G_B.$$  

Very interestingly, we found that this is also the sufficient condition since Eqs. (45) and (46) can be automatically satisfied once $G_A = G_B$.

Substituting Eq. (47) into Eqs. (21) and (29), the isotropic effective Young's modulus $E^{\text{eff}}$ can be obtained. In particular, if $v_A = 0.5$, $\Phi_B = 36.6\%$, it is found $E^{\text{eff}}/E_{\text{Voigt}} = 1.072$, which implies that the approximate Voigt estimation is not the upper bound of the effective isotropic Young's modulus. We also found that the
effective isotropic Young’s modulus cannot exceed the modulus of stiffest phase, i.e., \( \max(E_A, E_B) \).

4.2. The effective Young’s modulus for isotropic hierarchical composites

A more general and easier way to obtain two-phase isotropically elastic composites is adopting polycrystal strategy. Specifically, an isotropic hierarchical composite with randomly oriented layered microstructure at the bottom level can be constructed as shown in Fig. 7. A similar hierarchical structure has been observed in biocomposites, such as in shells [8]. Since we have already known the exact compliance matrix of the two-phase layered composite, i.e., the quasi-grain here, we may better estimate the Young’s modulus for isotropic hierarchical composites in Fig. 7.

Similar to previous section, the positive semi-definiteness of \( S_{\text{eff}} \) from Hill’s work [3] can be used to determine the lower bound of the Young’s modulus for isotropic hierarchical composites. Here, \( S_{\text{eff}} \) is the effective compliance matrix of the isotropic hierarchical composite, and \( S_{\text{grain}} \) is the weighted volume average of the compliance matrices of quasi-grains. Actually, the average is carried out over the orientation. By considering the random distribution of quasi-grains, it can be proved that

\[
S_{\text{eff}}^{11} = \frac{1}{5} \left( S_{11}^{\text{eff}} + S_{12}^{\text{eff}} + S_{13}^{\text{eff}} + \frac{2}{15} S_{12}^{\text{eff}} + S_{13}^{\text{eff}} + S_{23}^{\text{eff}} \right) + \frac{1}{15} S_{44}^{\text{eff}} + S_{55}^{\text{eff}} + S_{66}^{\text{eff}}.
\] (48)

Here, \( S_{ij}^{\text{eff}} \) are the compliance matrix components of a quasi-grain in its local principle coordinate system, and have been obtained in Section 2. Because of the positive semi-definiteness, the effective Young’s modulus for isotropic hierarchical composites \( E_{\text{eff}} \) satisfy the following inequality,

\[
S_{\text{eff}}^{11} - S_{\text{grain}}^{11} = S_{\text{eff}}^{11} - \frac{1}{5} S_{\text{grain}}^{11} \geq 0.
\] (49)

Therefore the lower bound of the isotropic Young’s modulus is

\[
\frac{E_{\text{lower}}}{E_A} = \frac{1}{5} \left( \frac{1}{5} + \frac{1}{5} + \frac{1}{5} + \frac{1}{5} + \frac{1}{5} \right) = \frac{1}{5}.
\]

\[
E_{\text{lower}} = \frac{1}{5} \left( S_{11}^{\text{eff}} + S_{12}^{\text{eff}} + S_{13}^{\text{eff}} + \frac{2}{15} S_{12}^{\text{eff}} + S_{13}^{\text{eff}} + S_{23}^{\text{eff}} \right) + \frac{1}{15} S_{44}^{\text{eff}} + S_{55}^{\text{eff}} + S_{66}^{\text{eff}}.
\] (50)

We now use the above equation to check the validity of Inequalities I and III (Eqs. (4) and (6)). For a hierarchical composite with \( V_A = 0.25, E_B/E_A = 0.9, \Phi_B = 50\% \), Fig. 8 shows the variation of the normalized lower bound for the Young’s modulus, \( E_{\text{lower}}/E_A \) as a function of the Poisson ratio of phase B, \( \nu_B \). Obviously, the isotropic effective Young’s modulus can exceed the approximate Voigt estimation \( E_{\text{Voigt}} \), which implies the invalidation of Inequality I \( (E_{\text{composite}} \leq E_{\text{Voigt}} = \Phi_A E_A + \Phi_B E_B) \). The maximum of the ratio \( E_{\text{lower}}/E_A \) can reach 1.103 when \( V_A = 0, \nu_B = 0.5, E_B/E_A = 0.918, \Phi_B = 43.9\% \). Moreover, it can also be found that Inequality III \( (E_{\text{composite}} \leq \max(E_A, E_B)) \) can be violated, e.g., \( E_{\text{lower}}/\max(E_A, E_B) = 1.102 \) when \( V_A = 0, \nu_B = 0.5, E_B/E_A = 1 \). It is interesting to note from Fig. 8 that the Reuss bound cannot be reached although it is still a lower bound.

5. Summary

The upper and lower bounds of the effective Young’s modulus with the Poisson effect are investigated in this paper. As an example, an effective compliance matrix for two-phase layered composites is accurately derived and the Poisson effect is taken into account. It is found that the effective Young’s modulus in both transverse and longitudinal direction can exceed not only the approximate Voigt estimation, but also the stiffness of the stiffer constituent phase. Moreover, the longitudinal (or parallel connection) Young’s modulus is not always larger than the transverse (or serial connection) one. For isotropic composites, it has also been demonstrated that the approximate Voigt estimation is not the upper bound for the effective Young’s modulus. Therefore, one should be careful in applying the well known bound estimations on the effective Young’s modulus of composites if one of the phases is near its incompressibility limit.

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