The equivalent axisymmetric model for Berkovich indenters in power-law hardening materials

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\textbf{A B S T R A C T}

Nix and Gio [Nix, W.D., Gao, H.J., 1998. Indentation size effects in crystalline materials: a law for strain gradient plasticity. Journal of the Mechanics and Physics of Solids 46, 411–425] established an important relation between the micro-indentation hardness and indentation depth for axisymmetric indenters. For the Berkovich indenter, however, this relation requires an equivalent cone angle. Qin et al. [Qin, J., Huang, Y., Xiao, J., Hwang, K.C., 2009. The equivalence of axisymmetric indentation model for three-dimensional indentation hardness. Journal of Materials Research 24, 776–783] showed that the widely used equivalent cone angle from the criterion of equal base area leads to significant errors in micro-indentation, and proposed a new equivalence of equal cone angle for iridium. It is shown in this paper that this new equivalence holds for a wide range of plastic work hardening materials. In addition, the prior equal-base-area criterion does not hold because the Berkovich indenter gives much higher density of geometrically necessary dislocations than axisymmetric indenter. The equivalence of equal cone angle, however, does not hold for Vickers indenter.

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1. Introduction

\cite{Nix98} established an axisymmetric model of indentation for sharp, conical indenters. They obtained a simple relation between the micro-indentation hardness $H$ and depth $h$

$$\left( \frac{H}{H_0} \right)^2 = 1 + \frac{h^*}{h},$$

(1)

where $H_0$ is the macro-indentation hardness for a large indentation depth,

$$h^* = \frac{27M^2}{2} \frac{b\nu^2}{\tan^2 \theta} \left( \frac{\mu}{H_0} \right)^2$$

(2)

is a characteristic length that depends on both the angle $\theta$ of the conical indenter (Fig. 1), Burgers vector $b$, shear modulus $\mu$, empirical coefficient $x$ around 0.3, and $M = 3.06$ for fcc metals. The linear relation between the square of micro-indentation hardness $H^2$ and the reciprocal of indentation depth $1/h$ agrees with the experimental data for single crystal and polycrystalline copper \cite{McElhaney98} and single crystal silver \cite{Ma95}. The above relation for
micro-indentation of conical indenters has been extended to spherical indenters (Swadener et al., 2002; Qu et al., 2006) or conical indenters with spherical tips (Xue et al., 2002; Qu et al., 2004) or different indenter angles (Qin et al., 2007), to nano-indentation (Huang et al., 2006, 2007), and to bcc metals (Qiu et al., 2001) and thin films on substrates (Saha et al., 2001; Zhang et al., 2007). There are other studies of indentation size effect (e.g., Begley and Hutchinson, 1998; Niordson and Hutchinson, 2003; Abu Al-Rub and Voyiadjis, 2004) based on strain gradient plasticity theories (e.g., Fleck and Hutchinson, 1993, 1997; Chen et al., 1999; Huang et al., 1999).

The above micro-indentation models assume axisymmetric indenters, but the Berkovich indenter in experiments, shown in Fig. 1, or Vickers indenter and cubic indenter (tip of a cube), are all non-axisymmetric. For the Berkovich indenter, the equivalent angle $\theta$ of the axisymmetric indenter is determined by assuming the same base area of contact at any indentation depth $h$. As illustrated in Fig. 1(a), the projected area of a regular triangular pyramid indenter with the indenter angle $\psi$ (i.e., triangle ABC) is the same as the projected area of the equivalent conical indenter (i.e., of circle P). This gives the indenter angle $\theta$ for the conical indenter as

$$\theta = \tan^{-1}\left(\frac{3\sqrt{3}}{\pi} \tan \psi\right).$$

(3)

For Berkovich indenter ($\psi = 65.3^\circ$), it gives the angle $\theta = 70.3^\circ$ of equivalent conical indenter. Three-dimensional finite element analysis shows that this equivalence holds approximately for the indentation load–displacement curve in macro-indentation hardness (Li et al., 2004), but not for contact area due to the indenter pile-up or sink-in effect (Qin et al., 2009). The difference in macro-indentation hardness (ratio of indentation load to contact area) given by the Berkovich indenter and axisymmetric indenter based on Eq. (1) is about 13% for copper. However, as shown in Fig. 2, this difference increases as the indentation depth decreases in micro-indentation (indentation depth on order of micrometer or sub-micrometer), and reaches 30% for the indentation depth of 1.5 $\mu$m in iridium (Qin et al., 2009).

For micro-indentation of iridium, Qin et al. (2009) established the equivalence of Berkovich and axisymmetric indenter by inscribing the axisymmetric cone to the triangular pyramid (Fig. 3), i.e.,

$$\theta = \psi.$$

(4)

The difference in micro-indentation hardness of iridium given by the Berkovich indenter and axisymmetric indenter based on Eq. (4) is only a few percent, as shown in Fig. 2.

The above equal-cone-angle criterion holds for iridium. The objective of this paper is to investigate the equivalence between the Berkovich indenter and axisymmetric indenter for a wide range of plastic work hardening materials. The conventional theory of mechanism-based strain gradient plasticity (CMSG) (Huang et al., 2004), which is based on the Taylor dislocation model (1934, 1938) and can account for the size effect in micro-indentation, is summarized in Section 2. The equivalence in Eq. (4) is then investigated in Section 3 for a large range of materials represented by different yield strength and plastic work hardening. Section 4 shows the effect of the round edges on the micro-indentation hardness, and also discusses the Vickers indenter.

2. The conventional theory of mechanism-based strain gradient plasticity (CMSG)

2.1. Taylor dislocation model

The shear flow stress $\tau$ is related to the dislocation density $\rho$ by $\tau = \mu b \sqrt{\rho}$ (Taylor, 1934, 1938), where $\mu$, $b$ and $x$ are the shear modulus, Burgers vector and empirical coefficient around 0.3, respectively. The dislocation density $\rho$ is composed of...
the densities $q_S$ for statistically stored dislocations (SSD) (Ashby, 1970) and $q_G$ for geometrically necessary dislocations (GND), (Nye, 1953; Cottrell, 1964; Ashby, 1970), i.e.,

$$q = q_S + q_G.$$ 

The GND density $q_G$ is related to the curvature of plastic deformation (Ashby, 1970; Nix and Gao, 1998), or effective plastic strain gradient $g_p$, by

$$q_G = \frac{1}{C^2} g_p$$

where $C^2 \approx 1.90$ is the Nye factor (Arsenlis and Parks, 1999) to reflect the effect of crystallography on the GND distribution, and $C^2$ is around 1.90 for fcc polycrystals (Arsenlis and Parks, 1999; Shi et al., 2004).

The tensile flow stress $r_{flow}$ is related to the shear flow stress $\tau$ by $r_{flow} = M\tau$, where $M$ is the Taylor factor, and $M = 3.06$ for fcc metals (Bishop and Hill, 1951a,b). The above relations yield $r_{flow} = MA\mu b\sqrt{q_s + \frac{r_d}{r^2}}$. For uniaxial tension, the flow stress is related to the plastic strain $\varepsilon^p$ by $r_{flow} = r_{ref}f(\varepsilon^p)$, where $r_{ref}$ is a reference stress and $f$ is a nondimensional function determined from the uniaxial stress–strain curve. Since the plastic strain gradient $\eta^p$ vanishes in uniaxial tension, the SSD density $q_S$ (Nix and Gao, 1998; Gao et al., 1999) is determined as $q_S = \left(\frac{r_{ref}}{\sigma_{ref}}\right)^2$. The above flow stress then becomes

$$r_{flow} = \sqrt{[r_{ref}f(\varepsilon^p)]^2 + M^2 r^2 \frac{\mu}{\sigma_{ref}} b \eta^p} = \sigma_{ref} \sqrt{f^2(\varepsilon^p) + b \eta^p},$$

where

$$l = M^2 r^2 \left(\frac{\mu}{\sigma_{ref}}\right)^2 b = 18 \alpha^2 \left(\frac{\mu}{\sigma_{ref}}\right)^2 b$$
is the intrinsic material length in strain gradient plasticity, which represents a natural combination of elasticity (shear modulus $\mu$), plasticity (reference stress $\sigma_{ref}$) and atomic spacing (Burgers vector $b$).

2.2. The constitutive model in CMSG

The volumetric strain rate $\dot{\varepsilon}_v = \frac{\sigma_v}{3\mu}$ and deviatoric strain rate \( \dot{\varepsilon}_d = \dot{\varepsilon}_d^e + \dot{\varepsilon}_d^p = \frac{\sigma_d}{2\mu} + \frac{\sigma_d^p}{\mu} \) in CMSG are related to the stress rate in the same way as in classical plasticity, where $K$ and $\mu$ are the elastic bulk and shear moduli, $\dot{\varepsilon}_v$ is the elastic deviatoric strain rate, and $\sigma_{ref} = (3\sigma_0/2)^{1/3}$ is the effective stress. The effective plastic strain rate $\dot{\varepsilon}_p = (2\dot{\varepsilon}_d^p/\sqrt{3})^{1/2}$ is related to the flow stress in Eq. (5) as \( \left( \frac{\sigma}{\sigma_{ref}} \right)^{1/2} \).

\[
\dot{\varepsilon}_p = \frac{\sigma^{m}}{\sigma_{ref}} \left( \frac{\sigma_e}{\sigma_{ref}} \right)^{1/2},
\]

where $\dot{\varepsilon} = (2\dot{\varepsilon}_d^p/\sqrt{3})^{1/2}$.

The stress rate is obtained in terms of the strain rate as

\[
\dot{\sigma}_{ij} = K\dot{\varepsilon}_{mn}\delta_{ij} + 2\mu \left\{ \frac{3\dot{\varepsilon}_d^e \sigma_{ij}^{\ref}}{2\sigma_e^2} \left[ \frac{\sigma_e}{\sigma_{ref} \sqrt{f(\sigma_e)} + \nu \sigma_e} \right]^{m} \dot{\varepsilon}_{kl} \right\},
\]

which is inverted to give the strain rate as

\[
\dot{\varepsilon}_e = \frac{1}{9K} \sigma_{mn} \delta_{ij} + \frac{1}{2\mu} \dot{\sigma}_d + \frac{\sigma_0}{2\mu} \left[ \frac{\sigma_e}{\sigma_{ref} \sqrt{f(\sigma_e)} + \nu \sigma_e} \right]^{m} \dot{\sigma}_e.
\]

Eq. (8) or (9) are identical to the constitutive relations for classical $J_2$-flow theory, $\dot{\varepsilon}_e = \frac{1}{9K} \sigma_{mn} \delta_{ij} + \frac{1}{2\mu} \dot{\sigma}_d + \frac{\sigma_0}{2\mu} \left[ \frac{\sigma_e}{\sigma_{ref} \sqrt{f(\sigma_e)} + \nu \sigma_e} \right]^{m}$, except that the incremental plastic modulus $h_p = \sigma_{ref}^2(\dot{\varepsilon}_e^p)$ in uniaxial tension is replaced by $3\mu \left\{ \frac{\sigma_{ref} \sqrt{f(\sigma_e)} + \nu \sigma_e}{\sigma_e} \right\}^{m} - 1$.

The effective plastic strain gradient $\eta^p$ is given by Huang et al. (2000, 2004)

\[
\eta^p = \int \dot{\eta}^p \, dt, \quad \dot{\eta}^p = \sqrt{\frac{1}{4} \dot{\varepsilon}_{ij}^p \dot{\varepsilon}_{ijk}^p}, \quad \dot{\varepsilon}_{ijkl} = \dot{\varepsilon}_{ijkl}^b + \dot{\varepsilon}_{ijkl}^e - \dot{\varepsilon}_{ijkl}^p,
\]

where $\dot{\varepsilon}_{ijkl}^p$ is the tensor of plastic strain rate. The equilibrium equations in CMSG are identical to those in conventional continuum theories. There are no extra boundary conditions beyond those in conventional continuum theories.

2.3. Finite element analysis of micro-indentation

The governing equations in CMSG are essentially the same as classical plasticity such that the existing finite element program can be easily modified to incorporate the plastic strain gradient effect. The constitutive relation of CMSG is implemented in the ABAQUS finite element program via its USER-MATERIAL subroutine UMAT. Its only difference from classical plasticity is that the plastic strain gradient must be evaluated in UMAT, which is accomplished by the numerical differentiation within the element, i.e., to interpolate the plastic strain increment $\Delta e^p$ within each element via the values at Gaussian integration points in the isoparametric space, and then to determine the gradient of plastic strain increment via the differentiation of the shape function.

Three-dimensional solid elements are used for the Berkovich indenter, while axisymmetric elements are used for the axisymmetric indenter with the equivalent angle \( \theta = \psi \). The finite sliding, hard contact model in ABAQUS allows sliding but no penetration between the rigid indenter and the indented material. The contact model also accounts for the indenter pile-up or sink-in. There are three symmetry planes for the Berkovich indenter such that only one sixth of the indenter is studied.

3. Equivalence of Berkovich and axisymmetric indenters for micro-indentation

The relation between stress and plastic strain in uniaxial tension can be written as

\[
\sigma = \sigma_0 + (C_p \eta^p)^N,
\]

where $\sigma_0$ is the yield stress, $N$ is the power-law hardening exponent and $C_p$ is a constant. For iridium, $\sigma_0 = 121$ MPa, $N = 0.638$ and $C_p = 222 \times 10^3$ (MPa)$^{0.638}$ (Qin et al., 2009). Other material properties include the Young’s modulus $E = 540$ GPa, Poisson’s ratio $\nu = 0.246$, Burgers vector $b = 0.271$ nm, the coefficient $\alpha = 0.430$ in the Taylor dislocation model and rate sensitivity exponent $m = 20$ (Qin et al., 2009). In the following, the power-law hardening exponent varies from $N = 0.638$ to 0.5 and
0.3, the coefficient changes from $C_p$ to $C_p/2$ and $C_p/4$, and the yield stress from $\sigma_0$ to $\sigma_0/2$ and $2\sigma_0$, to represent a wide range of plastic work hardening materials.

Fig. 4 shows the square of micro-indentation hardness $H^2$ versus the reciprocal of indentation depth $1/h$ for the Berkovich indenter (with sharp or round edges) and axisymmetric indenter with the same cone angle, $\theta = \psi$, for plastic work hardening exponent $N = 0.638$, 0.5 and 0.3.

Even though the curves for axisymmetric and Berkovich indenters seem to separate at small indentation depth, the relative errors are still very small. For $1/h = 0.85 \mu m^{-1}$, the relative errors in micro-indentation hardness are 2.6%, 0.6% and 0.05% for $N = 0.638$, 0.5 and 0.3, respectively. Therefore, the Berkovich indenter can be represented by the axisymmetric indenter with the same cone angle, $\theta = \psi$.

Figs. 5 and 6 also show the square of micro-indentation hardness $H^2$ versus the reciprocal of indentation depth $1/h$, but for the constant $C_p$, $C_p/2$ and $C_p/4$ and for the yield stress $\sigma_0/2$, $\sigma_0$ and $2\sigma_0$, respectively. Once again the indentation hardness for
the Berkovich indenter agrees well with that for the axisymmetric indenter for the entire range of micro-indentation. For example, for $1/h = 0.85 \mu m/C_0$, the relative error in micro-indentation hardness $2.6\%$ for $C_p$ is reduced to $1.3\%$ or less for

![Graph showing the square of micro-indentation hardness ($H^2$) versus the reciprocal of indentation depth ($1/h$) for the Berkovich indenter and axisymmetric indenter with the same cone angle, $\theta = \psi$, for yield stress $\sigma_0/2$, $\sigma_0$ and $2\sigma_0$.]

![Images of the density of geometrically necessary dislocations for Berkovich indenter (a and c) and axisymmetric indenter (b and d) at the indentation depth $h = 3.32 \mu m$.]

Fig. 6. The square of micro-indentation hardness ($H^2$) versus the reciprocal of indentation depth ($1/h$) for the Berkovich indenter and axisymmetric indenter with the same cone angle, $\theta = \psi$, for yield stress $\sigma_0/2$, $\sigma_0$ and $2\sigma_0$. 

the Berkovich indenter agrees well with that for the axisymmetric indenter for the entire range of micro-indentation. For example, for $1/h = 0.85 \mu m$, the relative error in micro-indentation hardness $2.6\%$ for $C_p$ is reduced to $1.3\%$ or less for
gives the micro-indentation hardness 3.73 GPa at the indentation depth.

Results validate the equivalence between the Berkovich indenter and axisymmetric indenter with the same cone angle ($\theta = 70.3^\circ$) shown in Fig. 7(d).

The Berkovich indenter in the present study is assumed to have sharp tip and edges. Qu et al. (2004) studied indenters with sharp tip and with round tip (tip radius of curvature 100 nm), and found them to give the same micro-indentation hardness. The micro-indentation hardness of Berkovich indenter with round edges of 100 nm radius of curvature is shown in Fig. 4, which is less than 0.6% different from that with sharp edges at the same indentation depth. Therefore, the Berkovich indenter with sharp tip and edges can also be represented by an axisymmetric indenter with the same cone angle.

The equivalence between Berkovich and axisymmetric indenters also holds for other triangular pyramid indenters (Qin et al., 2009). However, it may not hold for indenters of other shapes. We have used the finite element method described in Section 2.3 to study the micro-indentation hardness of Vickers indenter (square pyramid indenter with the cone angle $\psi = 68^\circ$). Only one eighth is studied due to the symmetry. The material is iridium with $\sigma_0 = 121$ MPa, $N = 0.638$ and $C_p = 222 \times 10^3$ (MPa)$^{1.0638}$ in Eq. (11), Young's modulus $E = 540$ GPa, Poisson's ratio $\nu = 0.246$, Burgers vector $b = 0.271$ nm, the coefficient $z = 0.430$ in the Taylor dislocation model and rate sensitivity exponent $m = 20$. The finite element method gives the micro-indentation hardness 3.73 GPa at the indentation depth $h = 2.1 \mu$m, while the axisymmetric indenter with the same cone angle ($\theta = 68^\circ$) gives the indentation hardness 4.05 GPa. As the indentation depth decreases to $h = 0.9 \mu$m, the difference between the Vickers and axisymmetric indenters increases to 15.2%, 4.95 and 5.70 GPa for the Vickers and axisymmetric indenters, respectively.

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